Dr. Mathias Braun & Wenhao Zhao



# Exercises for 'Functional Analysis 2' [MATH-404]

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**Ex 10.1 (Partial derivatives and $C^1$-functions)**
Let $(X_j)_{j=1}^n, Y$ be Banach spaces and $X = \prod_{j=1}^n X_j$ be equipped with the norm $\max_j \|x_j\|_{X_j}$. Let $U \subset \prod_{j=1}^n X_j$ be open and $F : U \to Y$. Show that if all partial derivatives exist and $\partial_{x_j} F \in C(U, \mathcal{L}(X_j, Y))$, then $F \in C^1(U, Y)$.
**Hint**: Guess the form of the derivative using a result in the lecture notes.

**Ex 10.2 (Consequences of Banach's fixed point theorem*)**
Let $X$ be a Banach space. Prove the following two statements :

a) Let $T : X \to X$. If there exists $\theta \in (0, 1)$ such that $\|T(x) - T(y)\| \le \theta \|x - y\|$ for all $x, y \in X$, then $I - T$ is a homeomorphism from $X$ to $X$.

b) Let $S : \overline{B_\delta(0)} \subset X \to X$ and assume that there exists $\theta \in (0, 1)$ such that
$$\|S(x) - S(y)\| \le \theta \|x - y\| \text{ for all } x, y \in \overline{B_\delta(0)}.$$
If $\|S(0)\| < \delta(1 - \theta)$, then $I + S$ has a unique zero. Moreover,
$$B_\rho(0) \subset (I + S)(B_\delta(0))$$
for $\rho = (1 - \theta)\delta - \|S(0)\|$.

**Ex 10.3 (Square root of an operator)**
Let $E$ be a Banach space and put $X = \mathcal{L}(E, E)$. Consider the function $F \colon X \to X$ such that $F(T) = T \circ T$ (which we can informally write as $F(T) = T^2$). Show that there exists a neighborhood $U$ of $I_E$ (the identity operator on $E$) and a differentiable map $G \colon U \to X$ such that $G(T)^2 = T$ for all $T \in U$.

**Ex 10.4 (Small-norm solutions of nonlinear BVP)**
Consider the nonlinear boundary value problem (BVP for short)
$$u'' + \lambda e^u = 0 \quad \text{in } (0, \pi), \quad u(0) = 0 = u(\pi).$$
Applying the implicit function theorem to the map $F \colon X \times \mathbb{R} \to Z$, $F(u, \lambda) = u'' + \lambda \exp(u)$, where
$$X := \left\{u \in C^2([0, \pi]) : \ u(0) = 0 = u(\pi)\right\} \quad \text{with norm } \|u\|_X = \sup_{t \in [0,\pi]} |u''(t)|,$$
$$Z := C([0, \pi]),$$

prove that for $\lambda$ in a neighborhood of 0 this problem has a unique small-norm solution that depends continuously on $\lambda$.

**Hint**: You can use without proof that $(X, \|\cdot\|_X)$ is a Banach space.

*Comment : in the above, the "small-norm" condition is specified, since otherwise uniqueness is not clear. Using convexity arguments, one could show that the solution is globally unique provided that* $\lambda \leq 0$ .